

On $H_3(1)$ Hankel determinant for some classes of univalent functions

K. O. BABALOLA^{1,2}

ABSTRACT. Focus in this paper is on the Hankel determinant, $H_3(1)$, for the well-known classes of bounded-turning, starlike and convex functions in the open unit disk $E = \{z \in \mathbb{C}: |z| < 1\}$. The results obtained complete the series of research works in the search for sharp upper bounds on $H_3(1)$ for each of these classes.

1. Introduction

Let A be the class of functions

$$f(z) = z + a_2 z^2 + \cdots \quad (1.1)$$

which are analytic in E . A function $f \in A$ is said to be of bounded turning, starlike and convex respectively if and only if, for $z \in E$, $\operatorname{Re} f'(z) > 0$, $\operatorname{Re} z f'(z)/f(z) > 0$ and $\operatorname{Re} (1 + z f''(z)/f'(z)) > 0$. By usual notations we denote these classes of functions respectively by R , S^* and C . Let $n \geq 0$ and $q \geq 1$, the q -th Hankel determinant is defined as:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

2000 *Mathematics Subject Classification.* 30C45; 30C50.

Key words and phrases. Hankel determinant, functions of bounded turning, starlike and convex univalent functions.

¹ Current Address: Centre for Advanced Studies in Mathematics, Lahore University of Management Sciences, Lahore, Pakistan. E-mail: kobabalola@lums.edu.pk

² Permanent Address: Department of Mathematics, University of Ilorin, Ilorin, Nigeria. E-mail: kobabalola@gmail.com; babalola.ko@unilorin.edu.ng

(see [11] for example). The determinant has been investigated by several authors with the subject of inquiry ranging from rate of growth of $H_q(n)$ as $n \rightarrow \infty$ [11, 12] to the determination of precise bounds on $H_q(n)$ for specific q and n for some favored classes of functions [4, 5, 10]. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles [4, 5, 10] for various classes of functions. In the present investigation, our focus is on the Hankel determinant, $H_3(1)$, for the well-known classes of bounded-turning, starlike and convex functions in E .

By definition, $H_3(1)$ is given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

For $f \in A$, $a_1 = 1$ so that

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

and by triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \quad (1.2)$$

Incidentally, all of the functionals on the right side of the inequality (1.2) have known (and sharp) upper bounds in the classes of functions which are of interest in this paper, except $|a_2a_3 - a_4|$. The last one is the well-known Fekete-Szegő functional. For R , sharp bound $2/3$ was reported in [1] (with R corresponding to $n = \alpha = 1, \beta = 0$ in the classes $T_n^\alpha(\beta)$ studied there) while for S^* and C , sharp bounds 1 and $1/3$ respectively were given in [6]. Janteng et-al [4, 5] obtained for the functional $|H_2(2)| \equiv |a_2a_4 - a_3^2|$ sharp bounds $4/9, 1$ and $1/8$ respectively for R, S^* and C . Furthermore, it is known that for $k = 2, 3, \dots$, $|a_k| \leq 2/k$, $|a_k| \leq k$ and $|a_k| \leq 1$ also respectively for R, S^* and C (see [2, 9]). Thus finding the best possible bounds on $|a_2a_3 - a_4|$ for each of the classes and using those known inequalities, then the sharp upper bounds on $H_3(1)$ follow as simple corollaries.

Our investigation follows a method of classical analysis devised by Libera and Zlotkiewicz [7, 8]. The same has been employed by many authors in similar works (see also [4, 5, 10]). In the next section we state the necessary lemmas while in Section 3 we present our main results.

2. Preliminary Lemmas

Let P denote the class of functions $p(z) = 1 + c_1z + c_2z^2 + \dots$ which are regular in E and satisfy $\operatorname{Re} p(z) > 0, z \in E$. To prove the main results in the next section we shall require the following two lemmas.

Lemma 2.1. ([2]) *Let $p \in P$, then $|c_k| \leq 2, k = 1, 2, \dots$, and the inequality is sharp.*

Lemma 2.2. ([7, 8]) *Let $p \in P$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \quad (2.1)$$

and

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2z(1 - |x|^2)(4 - c_1^2) \quad (2.2)$$

for some x, z such that $|x| \leq 1$ and $|z| \leq 1$.

3. Main Results

Theorem 3.1. *Let $f \in R$. Then*

$$|a_2a_3 - a_4| \leq \frac{1}{2}.$$

The inequality is sharp. Equality is attained by

$$f(z) = \int_0^z \frac{1+t^3}{1-t^3} dt.$$

Proof. Let $f \in R$. Then there exists a $p \in P$ such that $f'(z) = p(z)$, wherefrom equating coefficients we find that $2a_2 = c_1$, $3a_3 = c_2$ and $4a_4 = c_3$. Thus we have

$$|a_2a_3 - a_4| = \left| \frac{c_1c_2}{6} - \frac{c_3}{4} \right|. \quad (3.1)$$

Substituting for c_2 and c_3 using Lemma 2, we obtain

$$|a_2a_3 - a_4| = \left| \frac{c_1^3}{48} - \frac{c_1(4 - c_1^2)x}{24} + \frac{c_1(4 - c_1^2)x^2}{16} - \frac{(4 - c_1^2)(1 - |x|^2)z}{8} \right|. \quad (3.2)$$

By Lemma 1, $|c_1| \leq 2$. Then letting $c_1 = c$, we may assume without restriction that $c \in [-2, 0]$. Thus applying the triangle inequality on (3.2), with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{c^3}{48} + \frac{(4 - c^2)}{8} + \frac{c(4 - c^2)\rho}{24} + \frac{(c - 2)(4 - c^2)\rho^2}{16} \\ &= F(\rho). \end{aligned}$$

Now we have

$$F'(\rho) = \frac{c(4 - c^2)}{24} + \frac{(c - 2)(4 - c^2)\rho}{8} < 0.$$

Hence $F(\rho)$ is a decreasing function of ρ on the closed interval $[0, 1]$, so that $F(\rho) \leq F(0)$. That is

$$\begin{aligned} F(\rho) &\leq \frac{c^3}{48} + \frac{4 - c^2}{8} \\ &= G(c). \end{aligned}$$

Obviously $G(c)$ is increasing on $[-2, 0]$. Hence we have $G(c) \leq G(0) = 1/2$.

By setting $c_1 = c = 0$ and selecting $x = 0$ and $z = 1$ in (2.1) and (2.2) we find that $c_2 = 0$ and $c_3 = 2$. Thus equality is attained by $f(z)$ defined in theorem and the proof is complete. \square

Let $f \in R$. Then using the above result in (1.2) together with the known inequalities $|a_3 - a_2^2| \leq 2/3$ [1], $|a_2a_4 - a_3^2| \leq 4/9$ [4] and $|a_k| \leq 2/k$, $k = 2, 3, \dots$ [9], we have the sharp inequality:

Corollary 3.2. *Let $f \in R$. Then*

$$|H_3(1)| \leq \frac{993}{1620}.$$

Theorem 3.3. *Let $f \in S^*$. Then*

$$|a_2a_3 - a_4| \leq 2.$$

The inequality is sharp. Equality is attained by the Koebe function $k(z) = z/(1 - z)^2$.

Proof. Let $f \in S^*$. Then there exists a $p \in P$ such that $zf'(z) = f(z)p(z)$. Equating coefficients we find that $a_2 = c_1$, $2a_3 = c_2 + c_1^2$ and $6a_4 = 2c_3 + 3c_1c_2 + c_1^3$. Thus we have

$$|a_2a_3 - a_4| = \frac{1}{3}|c_1^3 - c_3|. \quad (3.3)$$

Substituting for c_3 from Lemma 2, we obtain

$$|a_2a_3 - a_4| = \frac{1}{12}|3c_1^3 - 2c_1(4 - c_1^2)x + c_1(4 - c_1^2)x^2 - 2(4 - c_1^2)(1 - |x|^2)z|. \quad (3.4)$$

Since $|c_1| \leq 2$ by Lemma 1, let $c_1 = c$ and assume without restriction that $c \in [0, 2]$. Applying the triangle inequality on (3.4), with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{1}{12}[3c^3 + 2(4 - c^2) + 2c(4 - c^2)\rho + (c - 2)(4 - c^2)\rho^2] \\ &= F(\rho). \end{aligned}$$

Differentiating $F(\rho)$, we have

$$F'(\rho) = \frac{1}{12}[2c(4 - c^2) + 2(c - 2)(4 - c^2)] > 0.$$

This implies that $F(\rho)$ is an increasing function of ρ on $[0, 1]$ if $c \in [1, 2]$. In this case $F(\rho) \leq F(1) = c \leq 2$ for all $\rho \in [0, 1]$. It follows therefore that $F(\rho) \leq 2$. On the other hand suppose $c \in [0, 1)$, then $F(\rho)$ is decreasing on $[0, 1]$ so that $F(\rho) \leq F(0)$. That is

$$\begin{aligned} F(\rho) &\leq \frac{3c^3 - 2c^2 + 8}{12} \\ &= G(c). \end{aligned}$$

Hence we have $G(c) \leq G(0) = 2/3$, $c \in [0, 1]$. This is less than 2, which is the case when $c \in [1, 2]$. Thus the maximum of the functional $|a_2a_3 - a_4|$ corresponds to $\rho = 1$ and $c = 2$.

If $c_1 = c = 2$ in (2.1) and (2.2), then we have $c_2 = c_3 = 2$. Using these in (3.3) we see that equality is attained which shows that our result is sharp. Furthermore, it is easily seen that the extremal function in this case is the well known Koebe function $k(z) = z/(1 - z)^2$. \square

For $f \in S^*$, using the known inequalities $|a_k| \leq k$, $k = 2, 3, \dots$ [2], $|a_2a_4 - a_3^2| \leq 1$ [5] and $|a_3 - a_2^2| \leq 1$ [6] together with Theorem 2 we have the next corollary.

Corollary 3.4. *Let $f \in S^*$. Then*

$$|H_3(1)| \leq 16.$$

The inequality is sharp. Equality is attained by a rotation, $k_1(z) = z/(1 + z)^2$, of the Koebe function.

Theorem 3.5. *Let $f \in C$. Then*

$$|a_2a_3 - a_4| \leq \frac{1}{6}.$$

The inequality is sharp. Equality is attained by

$$f(z) = \int_0^z \left\{ s \cdot \exp \left(\int_0^s \frac{2t^3}{1 - t^3} dt \right) \right\} ds.$$

Proof. For $f \in C$ given by (1.1), there exists a $p \in P$ such that $(zf'(z))' = f'(z)p(z)$. Then equating coefficients we find that $2a_2 = c_1$, $6a_3 = c_2 + c_1^2$ and $24a_4 = 2c_3 + 3c_1c_2 + c_1^3$. Thus we have

$$|a_2a_3 - a_4| = \frac{1}{24}|c_1^3 - c_1c_2 - 2c_3|. \quad (3.5)$$

Substituting for c_2 and c_3 using Lemma 2, we obtain

$$|a_2a_3 - a_4| = \frac{1}{48}|-3c_1(4 - c_1^2)x + c_1(4 - c_1^2)x^2 - 2(4 - c_1^2)(1 - |x|^2)z|. \quad (3.6)$$

With $|c_1| \leq 2$ from Lemma 1, we let $c_1 = c$ and assume also without restriction that $c \in [-2, 0]$. Thus applying the triangle inequality on (3.6), with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{(4 - c^2)}{24} + \frac{c(4 - c^2)\rho}{16} + \frac{(c - 2)(4 - c^2)\rho^2}{48} \\ &= F(\rho). \end{aligned}$$

Differentiating $F(\rho)$, we get

$$F'(\rho) = \frac{c(4 - c^2)}{16} + \frac{(c - 2)(4 - c^2)\rho}{24} < 0.$$

Thus $F(\rho)$ is a decreasing function of ρ on $[0, 1]$, so that $F(\rho) \leq F(0)$. That is

$$\begin{aligned} F(\rho) &\leq \frac{4 - c^2}{24} \\ &= G(c), \end{aligned}$$

which is increasing on $[-2, 0]$. Hence $G(c) \leq G(0) = 1/6$. Thus the maximum of the functional $|a_2a_3 - a_4|$ corresponds to $c = 0$ and $\rho = 0$, which is $1/6$.

If we set $c_1 = c = 0$ and selecting $x = 0$ and $z = 1$ in (2.1) and (2.2) we find that $c_2 = 0$ and $c_3 = 2$, and equality is attained by $f(z)$ defined in theorem. This completes the proof. \square

Finally for $f \in C$ if we use the known inequalities $|a_k| \leq 1$, $k = 2, 3, \dots$ [2], $|a_2a_4 - a_3^2| \leq 1/8$ [5] and $|a_3 - a_2^2| \leq 1/3$ [6] together with the last result, we obtain the following sharp inequality:

Corollary 3.6. *Let $f \in C$. Then*

$$|H_3(1)| \leq \frac{15}{24}.$$

Acknowledgements. This work was carried out at the Centre for Advanced Studies in Mathematics, CASM, Lahore University of Management Sciences, Lahore, Pakistan during the author's postdoctoral fellowship at the Centre. The author is indebted to all staff of CASM for their hospitality, most especially Prof. Ismat Beg.

REFERENCES

1. Babalola, K. O. and Opoola, T. O., *On the coefficients of certain analytic and univalent functions*, Advances in Inequalities for Series, (Edited by S. S. Dragomir and A. Sofo) Nova Science Publishers (<http://www.novapublishers.com>) (2008), 5-17.
2. Duren, P. L., *Univalent functions*, Springer Verlag. New York Inc. 1983.
3. Grenander, U. and Szego, G., *Toeplitz forms and their application*, University of California Press, Berkeley and Los Angeles, 1958.
4. Janteng, A., Abdulhalim, S. and Darus, M., *Coefficient inequality for a function whose derivative has positive real part*, J. Ineq. Pure and Applied Math. (<http://jipam.vu.edu.au/>) **7** (2) Art. 50 (2006), 1-5.
5. Janteng, A., Abdulhalim, S. and Darus, M., *Hankel determinant for starlike and convex functions*, Int. Journal of Math. Analysis, **1** (13) (2007), 619-625.
6. Keogh, F. R. and Merkes, E. P., *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20** (1969), 8-12.
7. Libera, R. J. and Zlotkiewicz, E. J., *Early coefficient of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85** (2) (1982), 225-230.
8. Libera, R. J. and Zlotkiewicz, E. J., *Coefficient bounds for the inverse of a function with derivative in P*, Proc. Amer. Math. Soc. **87** (2) (1983), 251-257.

9. Macgegor, T. H., *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc. **104** (1962), 532-537.
10. Mishra, A. K. and Gochhayat, P., *Second Hankel determinant for a class of analytic functions defined by fractional derivative*, Int. Journal of Math. and Math. Sciences, **2008** Art. 153280 (doi:10.1155/2008/153280), 1-10.
11. Noonan, J. W. and Thomas, D. K., *Second Hankel determinant of areally mean p -valent functions*, Trans. Amer. Math. Soc. **223** (1976), 337-346.
12. Noor, K. I., *Hankel determinant problem for the class of functions with bounded boundary rotation*, Rev. Roum. Math. Pures et Appl. **28** (8) (1983), 731-739.